Dimensional Reduction in Cobordism and K-theory [Blumenhagen, Cribiori, C.K., Makridou 220X.XXXXX]

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Cobordism - Definition

- Take two compact n-dimensional manifolds M and N
- Cobordant, if there is (n+1)-dimensional manifold W, s.t. $\partial W = M \cup N$

Figure 1: Cobordism W between M and N

Cobordism - Definition

- In physics terms: We allow for topology-changing processes between M and N
- We denote the set of cobordism classes Ω_n^ξ with elements [M] $\Rightarrow$ forms Abelian group under the disjoint union $[M_1] + [M_2] = [M_1 \cup M_2]$
- ξ denotes some kind of tangential structure, which is preserved on W, M and N

The Cobordism Conjecture [McNamara, Vafa '19]

- A non-vanishing cobordism group Ω_n^{ξ} can be understood as a global symmetry
  $\Rightarrow$ Absence of Global Symmetries in Quantum Gravity
  [Banks, Seiberg '10] implies $\Omega_n^{QG} = 0$

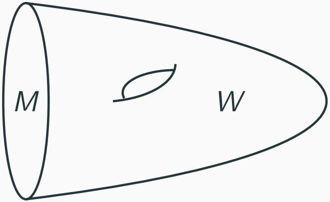

Figure 2: Vanishing cobordism group - nullbordant manifold

The Cobordism Conjecture

- Consider a simpler structure than the complete quantum gravity structure, but still expected part of a quantum gravity description (for example spin structure)
  $\Rightarrow$ non-trivial cobordism groups $\Omega_n^{\widetilde{QG}} \neq 0$
- Two possibilities to trivialize the global symmetry:
- Breaking: Introduce defect, such that
$$\Omega_n^{\widetilde{QG}+\text{defect}} = 0$$
- Gauging: Consistency of quantum gravity compactification on manifold M:
$$[M] = 0 \in \Omega_n^{\widetilde{QG}}$$

The Cobordism groups of general manifolds

- So far I have presented an abbreviated definition of cobordism groups, namely just $\Omega^{\xi}_n(pt) = \Omega^{\xi}_n$
- Just a special case of $\Omega^{\xi}_n(X)$
  $\Rightarrow$ set of equivalence classes [M,f] with $f : M \to X$
  (M,f) cobordant to (N,g), if there exists (W,h), such that:
  $\Rightarrow \partial W = M \cup N$
  $\Rightarrow$ h becomes f or g on the respective boundary

The Cobordism groups of general manifolds

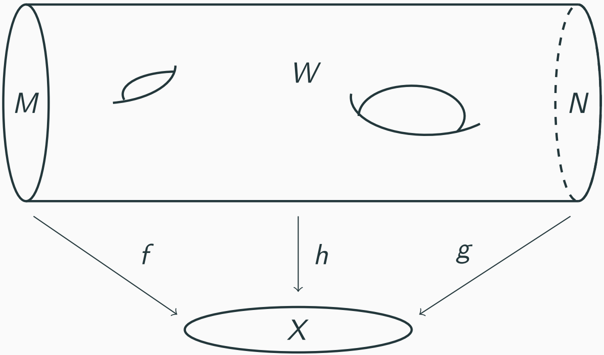

Figure 3: Cobordism (W, h) between (M, f) and (N, g).

What is K-Theory?

- Consider the pair of complex (or real) vector bundles (E, F)
  $\Rightarrow$ with equivalence relation $(E, F) \sim (E \oplus H, F \oplus H)$
- Can think about the pairing as subtraction, since
  $(E, E) \sim (0 \oplus E, 0 \oplus E) \sim (0, 0) = 0$ matching
  $(E, E) = E - E = 0$
- Can define addition, subtraction, the inverse of an element and the trivial element $\Rightarrow$ defines group
- K-theory is of physical relevance as well
  $\Rightarrow$ Classification of Dp-branes [Witten '98]

Connection between cobordism and K-theory

- for example real K-theory $KO^{-n}(pt)$ captures the D(9-n)-brane charges of type I:

n	0	1	2	3	4	5	6	7	8	9	10
$KO^{-n}(\mathrm{pt})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	D9	$\widehat{\mathrm{D8}}$	$\widehat{\mathrm{D7}}$		D5				D1	$\widehat{\mathrm{D0}}$	$\widehat{\mathrm{D(-1)}}$

- Now, take a look at the spin cobordism groups of the point:

n	0	1	2	3	4	5	6	7	8	9	10
$\Omega_n^{\mathrm{Spin}}(\mathrm{pt})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$2{\cdot}\mathbb{Z}$	$2{\cdot}\mathbb{Z}_2$	$3{\cdot}\mathbb{Z}_2$

- There is a remarkable similarity!

Connection between cobordism and K-theory

- This hints at a general structure ⇒ Atiyah-Bott-Shapiro orientation [Atiyah, Bott, Shapiro '64]:

$$\alpha: \quad \Omega_n^{\mathrm{Spin}}(\mathrm{pt}) \to KO^{-n}(\mathrm{pt}) \tag{1}$$

$$\alpha^c: \quad \Omega_n^{\mathrm{Spin}^c}(\mathrm{pt}) \to K^{-n}(\mathrm{pt}) \tag{2}$$

- α and α^c are the (refined) $\hat{A}$-genus and Todd-genus respectively
  ⇒ Can be explicitly calculated for a given manifold
- Recall: To gauge a non-vanishing cobordism group:

$$[M] \stackrel{!}{=} 0 \in \Omega_n^{\xi}$$

Connection between cobordism and K-theory

- Identified to match the tadpole constraint of a wrapped brane [Blumenhagen, Cribiori '21]
- Let's gauge $\Omega_6^{\mathrm{Spin}^c}(\mathrm{pt}) = \mathbb{Z} \oplus \mathbb{Z}$: $\Omega_6^{\mathrm{Spin}^c}(\mathrm{pt}) \to K^{-6}(\mathrm{pt})$

$$0 = \int_M \sum_{i \in \mathrm{def}} Q_i \, \delta^{(6)}(\Delta_{4,i}) + \int_M \left(a_1^{(6)} \frac{c_1 \, c_2}{24} + a_2^{(6)} \frac{c_1^3}{2} \right) .$$

- For $a_1^{(6)} = -12$ and $a_2^{(6)} = -30$, this tadpole is realized in F-theory on smooth CY_4 elliptically fibered over base M [Sethi, Vafa, Witten '96]

Generalizing from the point to a compact manifold X

- Dimensional reduction on $X \Rightarrow$ Calculation of $K^{-n}(X)$ required
- The cobordism conjecture states $\Omega_n(\mathrm{pt}) = 0$, but what about $\Omega_n(X)$?
- Is there a physical interpretation of the cobordism groups $\Omega_n(X)$?
- What happens to the map from cobordism to K-theory groups, when moving from pt to X?
- First, calculate cobordism with Spin/$\mathrm{Spin^c}$ structure and real/complex K-theory groups for several manifolds X of interest, e.g. $X \in \{S^k, T^k, K3, CY_3\}$
  $\Rightarrow$ Mathematical tool: Atiyah-Hirzebruch spectral sequence (AHSS) [Atiyah, Hirzebruch '61]

The Atiyah-Hirzebruch spectral sequence

- Applies both for generalized homology theories (like cobordism) and gen. cohomology theories (like K-theory)
- Choose X, for example CY_3, and the generalized (co-)homology theory of your interest, for example complex K-theory $K(X)$
- Starting point: second page $E_2^{p,q} = H^p(X, K^q(pt))$
- Goal: reach last page $E_\infty^{p,q}$ and sum over diagonals to get $K(X)$
- How to get from one page to the next?
  $\Rightarrow$ differentials d_r

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1} \qquad E_{r+1}^{p,q} \cong \frac{\ker d_r}{\mathrm{Im} d_r}$$

The Atiyah-Hirzebruch spectral sequence - Example

- Let's take a look at the AHSS for $K^{-n}(CY_3)$

6	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$
5	0	0	0	0	0	0	0
4	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$
3	0	0	0	0	0	0	0
2	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$
1	0	0	0	0	0	0	0
0	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$
-1	0	0	0	0	0	0	0
-2	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$
-3	0	0	0	0	0	0	0
-4	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$
-5	0 d_3	0	0	0 d_3	0	0	0
-6	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$

The Atiyah-Hirzebruch spectral sequence - Example II

- Differential d_3 has physical meaning!
- Enforces Freed-Witten anomaly cancellation, in our case fulfilled
  $\Rightarrow$ Let's compare results for $\Omega_n^{\mathrm{Spin}^c}(CY_3)$ and $K^{-n}(CY_3)$
- Is there still a connection between these two indicating gauging the global symmetries of $\Omega_n^{\mathrm{Spin}^c}(CY_3)$?
  $\Rightarrow$ the answer is yes (also true for the other manifolds we have studied)

Results of Spectral Sequence

- Interestingly, both $\Omega_n^{\mathrm{Spin^c}}(CY_3)$ and $K^{-n}(CY_3)$ can be expanded in terms of the (co-)homological data of CY_3:

$$\Omega_n^{\mathrm{Spin}^c}(CY_3) = \Omega_n^{\mathrm{Spin}^c}(\mathrm{pt}) \oplus b_2 \cdot \Omega_{n-2}^{\mathrm{Spin}^c}(\mathrm{pt}) \oplus b_3 \cdot \Omega_{n-3}^{\mathrm{Spin}^c}(\mathrm{pt}) \oplus b_4 \cdot \Omega_{n-4}^{\mathrm{Spin}^c}(\mathrm{pt}) \oplus \Omega_{n-6}^{\mathrm{Spin}^c}(\mathrm{pt})$$

$$K^{-n}(CY_3) = K^{-n}(\mathrm{pt}) \oplus b_2 \cdot K^{-n-2}(\mathrm{pt}) \oplus b_3 \cdot K^{-n-3}(\mathrm{pt}) \oplus b_4 \cdot K^{-n-4}(\mathrm{pt}) \oplus K^{-n-6}(\mathrm{pt})$$

- Lattice of global symmetries on the cobordism side $\Rightarrow$ as expected from dimensional reduction of the current associated to a (continuous) global symmetry

$$J_{n+m} = \sum_{a=1}^{b_m} j_n^{(m)a} \wedge \omega_{(m)a}$$

Results of Spectral Sequence

- Instead of just one tadpole condition from the ABS-map $\Omega_n^{\mathrm{Spin^c}}(\mathrm{pt}) \to K^{-n}(\mathrm{pt}) \Rightarrow$ Get a tadpole from gauging each cobordism group
  $\Rightarrow$ We can write down the following map indicating the gauging of all groups within $\Omega_n^{\mathrm{Spin}^c}(CY_3)$

$$\Omega_n^{\mathrm{Spin}^c}(CY_3) \to K_n(CY_3) = K^{-n+6}(CY_3)$$

- Gauging of each cobordism group introduces a tadpole corresponding on the K-theory side to a wrapped D-brane on a Calabi-Yau cycle

Conclusions & Outlook

- Fixing a specific manifold X in cobordism has a nice interpretation in terms of dimensional reduction
- The map from cobordism to K-theory is still intact and has a nice interpretation in terms of wrapped D-branes
- Considering manifolds with torsion
- So far H-flux has been turned off $\Rightarrow$ non-trivial H-flux studied for K-theory (twisted K-theory)

Thank You!